

Performance Evaluation of Quantum Merging: Negative Queue Length

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Abstract— We study the detail of queues handling quantum communication, especially so-called quantum merging of qubits. In quantum merging, by using EPR pairs (entangled pairs of quantum particles), jobs may bring extra service capacity to the server, which can be stored in the system for future use. We will model this phenomena as queues that can have the negative queue length, and show the basic characteristics of these quantum queues.

Keywords: quantum merging, negative entropy, negative customer, queue, performance evaluation, virtual waiting time

1 Introduction

Quantum information processing is fundamentally different with its classical counterpart. The difference can be used to achieve something that classical information processing cannot achieve. For example, Shor's algorithm[8] use qubits (quantum version of bits) to solve factorization problem effectively, which is notoriously hard to be solved by classical computers. This hardness is essential to the safety of RSA algorithm[7], that is used widely for secure communication on the internet.

On the other hand, sometimes quantum information processing shows us something strange to interpret. One of those examples, is entropy. In classical Shannon's theory, $H(X)$, the entropy of a random variable X , is defined by

$$H(X) = - \sum_x P\{X = x\} \log P\{X = x\}. \quad (1)$$

The Shannon's entropy measures the uncertainty of a data source, which is equivalent to how much information we can get when we learn the value of X . Suppose we have some partial data Y on our hand, which can be correlated with the source data X . Then, given Y , the amount of information we gained from X can be measured by the conditional entropy:

$$H(X|Y) = H(X, Y) - H(Y). \quad (2)$$

For example, if $H(X|Y) = 0$, then $X = f(Y)$, which means we already have complete information of X . Thus, intuitively (and mathematically provable), $H(X|Y)$ is always nonnegative. In the quan-

tum world, the Neumann's entropy $S(\rho)$ and conditional entropy $S(\rho|\sigma)$ are defined by

$$S(\rho) = -tr(\rho \log \rho), \quad (3)$$

$$S(\rho|\sigma) = S(\rho, \sigma) - S(\sigma), \quad (4)$$

for the density operators of quantum states ρ and σ . Although the Neumann's entropy is a natural extension of Shannon's entropy and measures the information carried, $S(\rho)$ can be negative for EPR pairs (p.514 in [6]). In other words, we have the information more than certain for EPR pairs, which has been the mystery of quantum information theory.

In 2005, Horodecki, Oppenheim and Andreas proposed the concept of partial information in [5]. They also introduced the framework called quantum merging to solve the mystery of negative quantum entropy. They showed that in the case when $S(\rho) < 0$, they can store the excess of the certainty in the bank and use them in the future. Thus, the performance of quantum merging is certainly different with the classical information processing. In this paper, we discuss the performance of the queue for quantum merging in detail, using queues that can have the negative queue length.

In 1991, Gelenbe introduced the concept of negative customers in Markovian network setting[3]. There, a negative customers are supposed to erase one positive customer in queue while the queue length is positive. The stability and product form solution are studied in the papers like [3, 2, 11]. However, the negative customers in our model can be kept in the queue for the future use, even when there is no

positive customers. That is the main new concept derived from quantum merging.

2 Quantum Communication by Qubit Merging

We introduce the framework of quantum merging proposed in [5]. Suppose Alice wants to transfer the state of qubits to Bob. But in this case, Bob has his qubit, which is probably correlated with the state of Alice's. In other words, Alice wants to send some information to Bob and merge her state to Bob's. Let ρ_A and ρ_B be Alice and Bob's density operator and let ρ_{AB} be the joint quantum state of Alice and Bob. Alice can use both the quantum channel and classical channel. Depending on the Bob's quantum state, the procedure is different. Let us consider the following three cases. To distinguish the bits, we use $|\cdot\rangle_A$ for the Alice's and $|\cdot\rangle_B$ for Bob's bit.

1. Independent states ($S(\rho_A|\rho_B) = 1$). In this case, Alice has a random bit:

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A), \quad (5)$$

and Bob has his some state:

$$\rho_B = |0\rangle_B. \quad (6)$$

The joint density matrix is

$$\rho_{AB} = \frac{1}{2}(|0\rangle\langle 0|_A + |1\rangle\langle 1|_A)|0\rangle_B. \quad (7)$$

In this case, Alice has to send one qubit with the quantum channel to merge her state.

2. Classically strongly-correlated states ($S(\rho_A|\rho_B) = 0$). In this case the joint quantum state is

$$\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00|_{AB} + |11\rangle\langle 11|_{AB}). \quad (8)$$

The mixed state can be considered a part of pure state introducing a reference system R . The state is already synchronized. Thus, Alice doesn't have to send any quantum information. Alice need to do some measurement (maybe for a reference system), then she can use classical channel to send the result to Bob, then Bob can adjust his state locally to merge Alice's state.

3. Entangled state ($S(\rho_A|\rho_B) = -1$). The joint quantum state is a pure and entangled state, for example,

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (9)$$

In this case, this EPR state is not only synchronized, but also has the ability to transfer other qubit (as a tool of quantum teleporation), while Bob creates a new EPR pair locally.

Thus, it is clear that in quantum merging, the arrival of jobs (sending qubits) is not necessary a workload, but also enhance the service capacity, which is radically different from the classical bit merging.

3 Queues with Negative Queue Length

We model the quantum merging scheme and evaluate its performance using queueing theory.

3.1 Queuing Model and Negative Virtual Waiting Time

Suppose Alice and Bob build up a system for a quantum merging and open it for public. Customers arrive at the system carrying a qubit to merge to the receiver end. For the simplicity, we assume two types of customers:(1) whose qubit is independent with the receiver end, (2) whose qubit is entangled with the qubit at the receiver end. The first is called positive customers and the later is called negative customers. The both positive and negative customers arrive at the system with independent Poisson processes with the rate λ_p and λ_n , respectively. We can assume another type of customers whose qubit is classically strongly-correlated. However, those customers doesn't have to use quantum communication, so we ignore here.

As showed in the Section 2, the positive customer has to send his state to the receiver with a quantum channel, which is required some service time S . Maybe this will be done by using a photonic cable by sending a new EPR pair. The service time S is assumed to be independent and identically distributed with $G(x) = P\{S \leq x\}$. On the other hand, a negative customer has a EPR pair, which can be used for sending the quantum state of a positive customer by using the quantum teleporation. We assume whenever there is a positive customer waiting, those EPR pair of the negative customers will be used. When there is no positive customer in the system, we can save EPR pairs for future use. The queueing discipline is assumed to be first-come-first-serve. Let $N(t)$ be the number of customers in the system. We allow $N(t)$ to be negative. When the queue length $N(t)$ is negative, we have a pool of EPR pairs or negative customers in the system. The queue can be called an $M/G/1\pm$ queue for indicating to have both positive and negative queue length.

Here is the summary of the system transitions:

1. When a positive customer sees $N(t) < 0$ at his arrival, then he consumes one negative customer to send his qubit and leaves the system immediately.
2. When a positive customer sees $N(t) \geq 0$, then he joins the queue and waits for his service. His service will be terminated by the normal service completion required S , or by an arrival of negative customer.

3. When a negative customer sees $N(t) > 0$, then the customer in service finishes the service immediately and leave the system.
4. When a negative customer sees $N(t) \leq 0$, then he will join the (negative) queue.

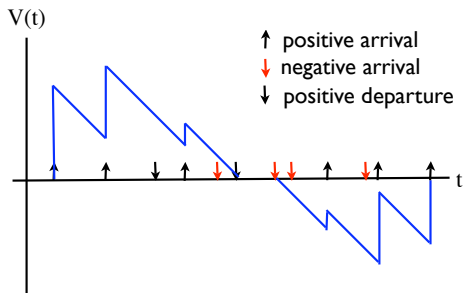


Figure 1: Virtual waiting time in a queue with negative queue length.

Now we extend the definition of the virtual waiting time $V(t)$ for the queues with negative queue length. Suppose an additional virtual positive customer has been arrived at our system at the time t . We allow this customer to arrive prior to the time t . The virtual waiting time $V(t)$ is the difference between t and the time when this virtual customer starts his service. When the queue length is positive, the start time of his service is later than t , so $V(t)$ is the same as the ordinary virtual waiting time. However, when the queue length is negative, this virtual customer could have started his service at the arrival of the negative customer who arrived earliest in the queue. Thus, $V(t)$ can be both negative and positive. In Figure 1, we depict a sample path of our virtual waiting time process.

We will evaluate the mean of the stationary version of the virtual waiting time $E[V(t)]$ by censoring the virtual waiting time process.

3.2 Censoring and Idle-time Modification

In order to use ordinary queuing theory, we will break up our virtual waiting time process into pieces. Let us extend the notion of busy period in the case of $V(t) < 0$. Thus, when the system is idle, both negative and positive customers can start a new busy period. Since the positive and negative arrivals are Poisson process, the starts of new positive busy period are renewal points. We will pick only busy cycles started by negative customers and put them into one process (see Figure 2). Let $V_n(t)$ be the virtual waiting time of this censored process.

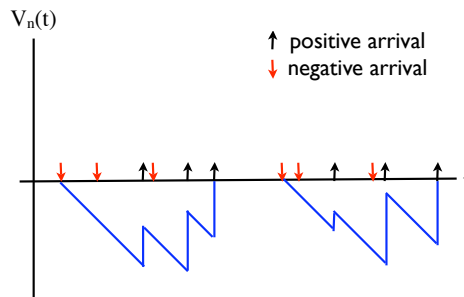


Figure 2: $V_n(t)$: Virtual waiting time in the negatively censored process.

it is easy to see that $-V_n(t)$ is nothing but the attained sojourn time process. The attained sojourn time is probabilistically equivalent with the virtual waiting time, especially, the length of corresponding busy period length is the same [9]. Moreover, this negatively censored process has the same probabilistic structure with $M/M/1$ queues except the idle period. Note that idle periods are exponentially distributed with the mean $1/(\lambda_p + \lambda_n)$, not $1/\lambda_n$.

Now we will modify the idle periods. We will take out all the idle period, then substitute them with new exponential random variables with the mean $1/\lambda_n$, which is independent with others. Note that this modification will not affect the sample path of virtual waiting time in the busy period. The resulted modified process is an $M/M/1$ queue with the arrival rate λ_n and the service rate λ_p . Let Y_n be the busy period of this modified process, then given $\lambda_p > \lambda_n$, we have

$$E[Y_n] = \frac{1}{\lambda_p - \lambda_n}, \quad (10)$$

(for example see [10, 4]). Further, let $W_n(t)$ be the stationary version of the virtual waiting time of the modified process, then

$$E[W_n(t)] = \frac{\lambda_n/\lambda_p}{\lambda_p - \lambda_n}. \quad (11)$$

By adjusting the idle period replacement, we can obtain the mean virtual waiting time of negative process, $V_n(t)$ as

$$\begin{aligned} E[-V_n(t)] &= \frac{E[Y_n] + 1/\lambda_n}{E[Y_n] + 1/(\lambda_p + \lambda_n)} E[W_n(t)] \\ &= \frac{\lambda_n + \lambda_p}{2\lambda_p(\lambda_p - \lambda_n)}. \end{aligned} \quad (12)$$

Now we consider the positive part. Just like the negative busy cycles, we will take out the positive

cycles and put them into one process. Let $V_p(t)$ be the the virtual waiting time of this positive process. See Figure 3

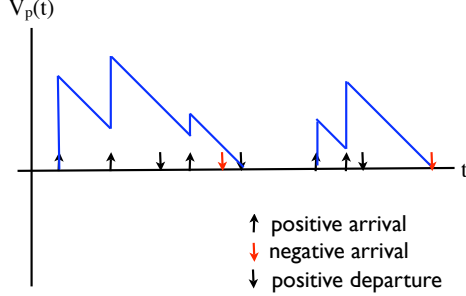


Figure 3: $V_p(t)$: Virtual waiting time in the positively censored process.

Next we modify all idle period with new exponential random variables with the mean $1/\lambda_p$. The resulted modified process is an ordinary $M/G/1$ queue with the arrival rate λ_p and the i.i.d. service time S_p defined by

$$S_p = \min(S, T_n), \quad (13)$$

where T_n is the time length to the next negative arrival and is the exponential random variable with its mean λ_n .

Let $\mu_p = 1/E[S_p]$, Y_p be the length of busy period in this modified $M/G/1$ queue. Then,

$$E[Y_p] = \frac{1}{\mu_p - \lambda_p}. \quad (14)$$

Let W_p be the virtual waiting time of the modified $M/G/1$ queue, then given $\lambda_p < \mu_p$, we have

$$E[W_p] = \frac{\lambda_p E[S_p^2]/2}{1 - \lambda_p/\mu_p}, \quad (15)$$

by Pollaczek-Khinchine formula (see [10, 4]). Adjusting the replacement of the idle time, we have

$$\begin{aligned} E[V_p(t)] &= \frac{E[Y_p] + 1/\lambda_p}{E[Y_p] + 1/(\lambda_p + \lambda_n)} E[W_p] \\ &= \frac{(\lambda_p + \lambda_n)E[S_p^2]/2}{(1 + \lambda_n/\mu_p)(1 + \lambda_p/\mu_p)}. \end{aligned} \quad (16)$$

3.3 Busy Cycle that Arrivals See

Let P_p be the probability that a customer sees a positive busy cycle, and P_n be the probability that a customer sees a negative one. At the start of the busy period, a positive customer starts busy period with the probability $\lambda_p/(\lambda_p + \lambda_n)$. Since the arrival

of customers is assumed to be a Poisson process, we have

$$P_p = \frac{(\lambda_p - \lambda_n)(\mu_p + \lambda_n)}{(\lambda_p + \lambda_n)(\mu_p - \lambda_n)}. \quad (17)$$

Also,

$$\begin{aligned} P_n &= 1 - P_p \\ &= \frac{2\lambda_n(\mu_p + \lambda_n)}{(\lambda_p + \lambda_n)(\mu_p - \lambda_n)}. \end{aligned} \quad (18)$$

Note that these probabilities are insensitive with the second moment of the service time S_p . It is easy to see that the condition $\lambda_p < \mu_p$ will guarantee $P_p < 1$, since we have always $\mu_p = 1/E[\min(S, T_n)] > \lambda_n$.

When the arrival rate of negative customers gets bigger and $\lambda_n \rightarrow \lambda_p$, we have $P_p \rightarrow 0$ and $P_n \rightarrow 1$ as expected. On the other hand, when the arrival rate of positive customers approaches to the service rate μ_p , we have $P_p \rightarrow 1$ and $P_n \rightarrow 0$.

3.4 Virtual Waiting Time

By conditioning on the busy cycle that a customer sees, we have

$$\begin{aligned} E[V(t)] &= E[V(t)|\text{an arrival sees a positive cycle}]P_p \\ &\quad + E[V(t)|\text{an arrival sees a negative cycle}]P_n. \end{aligned} \quad (19)$$

Given that an arrival sees a positive cycle, $V(t)$ is equivalent to $V_p(t)$ in distribution. Thus, by using (16), we have

$$\begin{aligned} E[V(t)|\text{an arrival sees a positive cycle}] &= E[V_p(t)] \\ &= \frac{(\lambda_p + \lambda_n)E[S_p^2]/2}{(1 + \lambda_n/\mu_p)(1 + \lambda_p/\mu_p)}. \end{aligned} \quad (20)$$

Similarly, by using (12)

$$\begin{aligned} E[V(t)|\text{an arrival sees a negative cycle}] &= E[V_n(t)] \\ &= -\frac{\lambda_n + \lambda_p}{2\lambda_p(\lambda_p - \lambda_n)}. \end{aligned} \quad (21)$$

Hence, we have the mean virtual waiting time of this system:

$$\begin{aligned} E[V(t)] &= \frac{\mu_p^2(\lambda_p - \lambda_n)E[S_p^2]/2}{(\mu_p - \lambda_n)(\mu_p - \lambda_p)} \\ &\quad - \frac{\lambda_n(\mu_p - \lambda_p)}{\lambda_p(\mu_p - \lambda_n)(\lambda_p - \lambda_n)}. \end{aligned} \quad (22)$$

Note that when we have no negative customers, $\lambda_n = 0$ and $E[S_p^2] = E[S^2]$ in (22). Thus, we have

$$E[V(t)] = \frac{\lambda_p E[S^2]/2}{1 - \lambda_p E[S]}. \quad (23)$$

which is the usual Pollaczek-Khinchin formula for $M/G/1$ queues.

Let W be the the ordinary waiting time for customers in the stationary state. Note that negative customers always see no waiting time. Since the arrival is Poisson, $W = \max(V(t), 0)$ in distribution. Thus,

$$\begin{aligned} E[W] &= E[V_p(t)]P_p \frac{\lambda_p}{\lambda_p + \lambda_n} \\ &= \frac{\lambda_p \mu_p^2 (\lambda_p - \lambda_n) E[S_p^2]/2}{(\mu_p - \lambda_n)(\mu_p - \lambda_p)(\lambda_p + \lambda_n)}, \end{aligned} \quad (24)$$

where $\lambda_n < \lambda_p < \mu_p$.

4 Example of an $M/D/1\pm$ queue

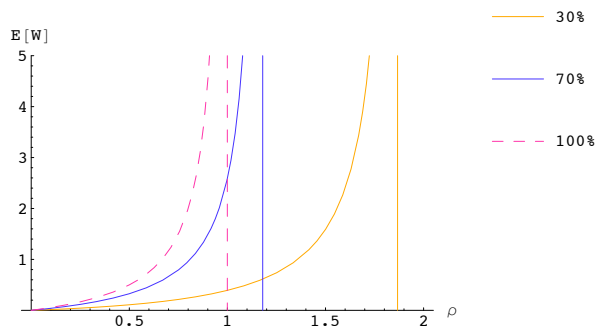


Figure 4: Mean waiting time in the $M/D/1\pm$ queue. The percentage shows the ratio of positive customers in the arrival customers. The 100% corresponds to the ordinary $M/D/1$ queue.

Let us consider an $M/D/1\pm$ queue. The service time of customers S is constant, that is, $S = 1/\mu$. In this case, $1/\mu_p = E[S_p]$ and $E[S_p^2]$ in (24) can be calculated by

$$E[S_p] = E[\min(S, T_n)] = \frac{1}{\lambda_n} \{1 - e^{-\lambda_n/\mu}\}. \quad (25)$$

and

$$E[S_p^2] = \frac{2}{\lambda_n^2} - \frac{2(\mu + \lambda_n)e^{-\lambda_n/\mu}}{\lambda_n^2 \mu}. \quad (26)$$

Figure 4 shows the mean waiting time of $M/D/1\pm$ queues with $S = 1$. Unlike the ordinary $M/D/1$ queue, the mean waiting time will not blow up for $\rho = \lambda/\mu > 1$. We can see “positive” effect of negative customers here.

5 Conclusion

We develop a queueing model for quantum merging. The queueing model we proposed is the $M/G/1\pm$ queue that can have the negative queue length and negative virtual waiting time. We derive the explicit form of the mean waiting time of as well as the mean virtual waiting time in this $M/G/1\pm$ queue.

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